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EE 359: Digital Signal Processing
11/14/2025

Homework 7

PROBLEM 8.21

(a) We want a sequence $\tilde{y}_1[n]$ whose DFS is

$$\tilde{Y}_1[k] = \tilde{X}_1[k]\tilde{X}_2[k].$$

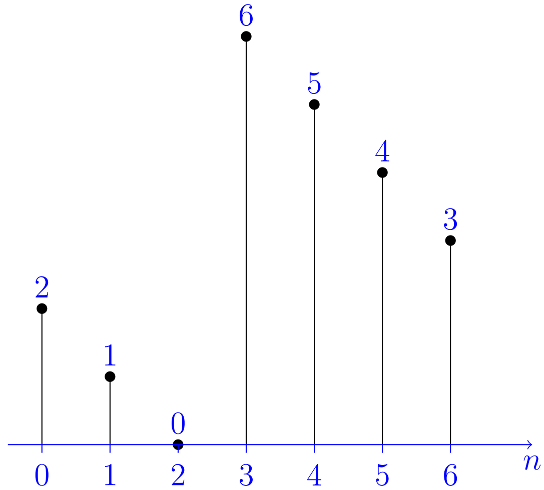
For $N = 7$, multiplication in the DFS domain corresponds to periodic convolution:

$$\tilde{y}_1[n] = \sum_{m=0}^{6} \tilde{x}_1[m]\tilde{x}_2[n-m].$$

The signal $\tilde{x}_2[n]$ is a periodic impulse at $n \equiv 2 \pmod 7$, so convolution shifts $\tilde{x}_1[n]$ by 2. Thus

$$\tilde{y}_1[n] = \tilde{x}_1[n-2].$$

A stem plot for one period, with labels, is shown below.



(b) The DFS of $\tilde{x}_3[n]$ is

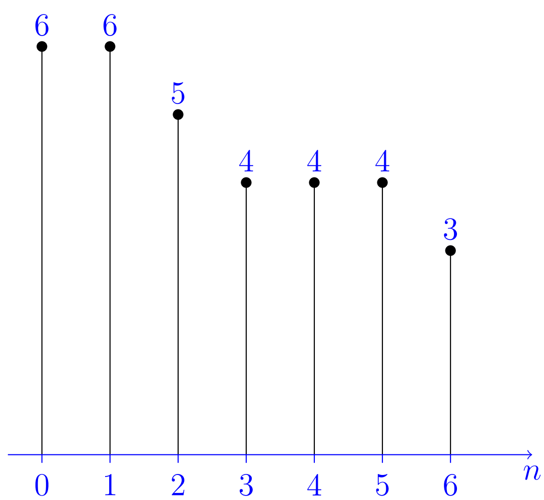
$$\tilde{X}_3[k] = 1 + W_7^{4k}.$$

Then

$$\tilde{Y}_2[k] = \tilde{X}_1[k]\tilde{X}_3[k] = \tilde{X}_1[k] + W_7^{4k}\tilde{X}_1[k].$$

Multiplying by W_7^{4k} corresponds to shifting by 4, so

$$\tilde{y}_2[n] = \tilde{x}_1[n] + \tilde{x}_1[n-4].$$



Problem 8.23

The six–point sequence $x[n]$ from Fig. P23 is

$$x[0] = 4, \quad x[1] = 3, \quad x[2] = 2, \quad x[3] = 1,$$

and all other samples (up to $n = 5$) are zero.

(a). We are told that

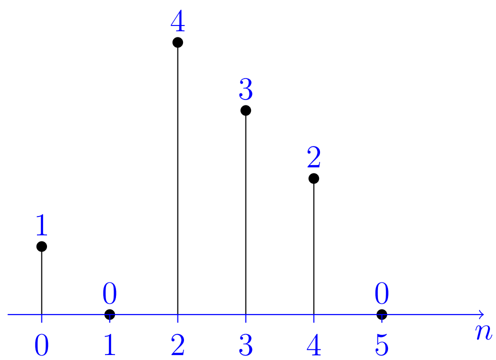
$$Y[k] = W_6^{5k} X[k],$$

which is a circular time–shift property. Since multiplying by W_6^{5k} corresponds to a shift by +5 samples (modulo 6), the sequence is

$$y[n] = x[n-5].$$

Thus $y[n]$ is $x[n]$ shifted to the right by 5, or equivalently by −1.

$$y[0] = 1, \; y[1] = 0, \; y[2] = 4, \; y[3] = 3, \; y[4] = 2, \; y[5] = 0.$$



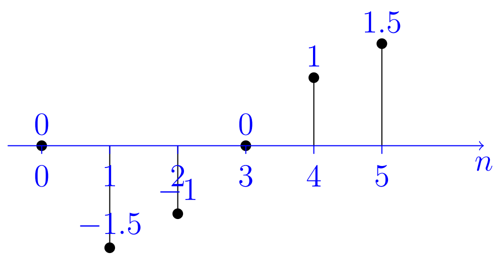
(b). Now

$$W[k] = \text{Im}\{X[k]\}.$$

The inverse six–point DFT gives

$$w[n] = \text{Im}\{x_{\text{sym}}[n]\} = \{\, 0, \, -\tfrac{3}{2}, \, -j, \, 0, \, j, \, \tfrac{3j}{2} \,\},$$

with indices $n = 0, \ldots, 5$.



(c). Given

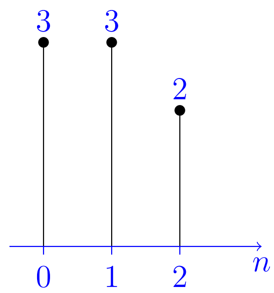
$$Q[k] = X[2k+1], \qquad k = 0, 1, 2,$$

which samples $X[k]$ at odd indices. The computation in the text gives

$$Q[k] = 3 + 3e^{-j\pi k/3} + 2e^{-j2\pi k/3}.$$

The inverse three–point DFT results in the sequence

$$q[0] = 3, \qquad q[1] = 3e^{-j\pi/3}, \qquad q[2] = 2e^{-j2\pi/3}.$$



Problem 8.31

Given

$$x[n] = 2\delta[n] + \delta[n-1] - \delta[n-2].$$

(a). The DTFT of $x[n]$ is

$$X(e^{j\omega}) = 2 + e^{-j\omega} - e^{-j2\omega}.$$

If $y[n] = x[-n]$, then by the time–reversal property

$$Y(e^{j\omega}) = X(e^{-j\omega}) = 2 + e^{j\omega} - e^{j2\omega}.$$

(b). We now form

$$W(e^{j\omega}) = X(e^{j\omega})Y(e^{j\omega}) = (2 + e^{-j\omega} - e^{-j2\omega})(2 + e^{j\omega} - e^{j2\omega}).$$

Multiplying the two factors gives

$$\begin{aligned} W(e^{j\omega}) &= 4 + 2e^{-j\omega} - 2e^{-j2\omega} + 2e^{j\omega} + 1 - e^{-j\omega} - 2e^{j2\omega} - e^{j\omega} + e^{-j2\omega} \\ &= -2e^{j2\omega} + e^{j\omega} + 6 + e^{-j\omega} - 2e^{-j2\omega}. \end{aligned}$$

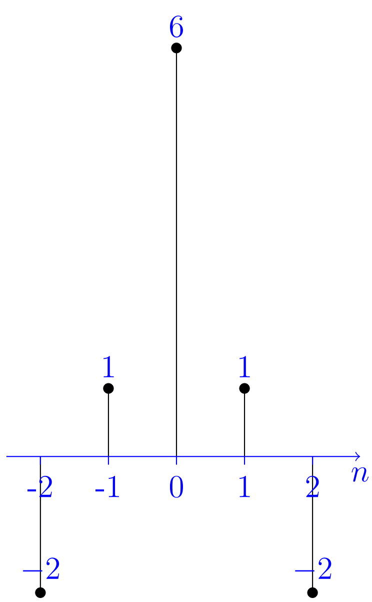
(c). From the form of $W(e^{j\omega})$ we can see $w[n] = x[n] * y[n]$:

$$w[n] = -2\delta[n+2] + \delta[n+1] + 6\delta[n] + \delta[n-1] - 2\delta[n-2].$$

So

$$w[-2] = -2, \quad w[-1] = 1, \quad w[0] = 6, \quad w[1] = 1, \quad w[2] = -2,$$

and $w[n] = 0$ otherwise.



(d). Now define

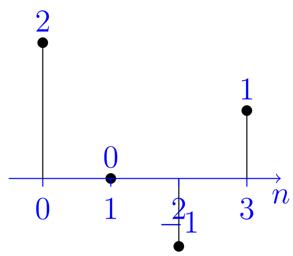
$$y_p[n] = x[((-n))_4], \qquad 0 \le n \le 3,$$

where $((\cdot))_4$ means modulo 4. We first write $x[n]$ over one period of length 4:

$$x[0] = 2, \quad x[1] = 1, \quad x[2] = -1, \quad x[3] = 0.$$

Then

$$\begin{aligned} y_p[0] &= x[0] = 2, \\ y_p[1] &= x[(-1)_4] = x[3] = 0, \\ y_p[2] &= x[(-2)_4] = x[2] = -1, \\ y_p[3] &= x[(-3)_4] = x[1] = 1. \end{aligned}$$



(e). We now form the 4–point circular convolution of $x[n]$ with $y_p[n]$:

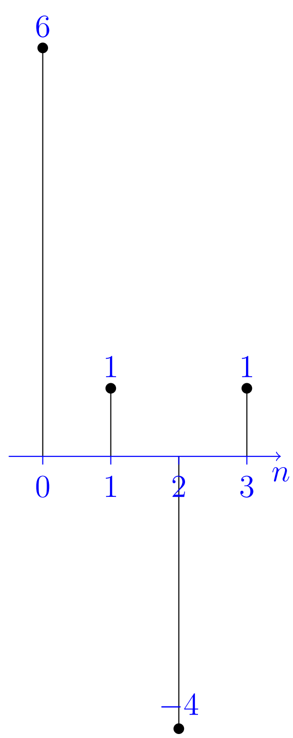
$$w_p[n] = \sum_{m=0}^{3} x[m]\, y_p[((n-m))_4], \qquad n = 0, 1, 2, 3.$$

Using $x = [2, 1, -1, 0]$ and $y_p = [2, 0, -1, 1]$:

$$\begin{aligned} w_p[0] &= 2 \cdot 2 + 1 \cdot 1 + (-1) \cdot 0 + 0 \cdot (-1) = 6, \\ w_p[1] &= 2 \cdot 0 + 1 \cdot 2 + (-1) \cdot 1 + 0 \cdot 0 = 1, \\ w_p[2] &= 2 \cdot (-1) + 1 \cdot 0 + (-1) \cdot 2 + 0 \cdot 1 = -4, \\ w_p[3] &= 2 \cdot 1 + 1 \cdot (-1) + (-1) \cdot 0 + 0 \cdot 2 = 1. \end{aligned}$$

So

$$w_p[0] = 6, \quad w_p[1] = 1, \quad w_p[2] = -4, \quad w_p[3] = 1.$$



Note that $w_p[n]$ is periodic with period 4.

(f). The linear convolution $w[n] = x[n] * x[-n]$ in part (c) has nonzero values for

$$-2 \le n \le 2,$$

so its length is 5. To avoid time–domain aliasing, the circular convolution length N must be at least the length of the linear convolution. Therefore

$$N \ge 5.$$

For any $N \ge 5$, the circular convolution of $x[n]$ with $x[((-n))_N]$ will be identical to $w[n]$ from part (c).

Problem 8.43

(a) The 1024-point DFT of $x[n]$ is $X[k]$. The sequence $R[k]$ is obtained by compressing $X[k]$ by 2, so

$$R[k] = X[2k], \qquad k = 0, \dots, 511.$$

Then $r[n]$ is the 512-point IDFT of $R[k]$. Compressing the spectrum by 2 means we undersample $X(e^{j\omega})$ in frequency, which causes aliasing in time. In this case the second half of $x[n]$ (samples 512–1023) folds onto the first half. Thus

$$r[n] = x[n] + x[n + 512], \qquad 0 \le n \le 511,$$

and $r[n] = 0$ outside $0 \le n \le 511$. This corresponds to choice (iii).

(b) Now $Y[k]$ is obtained by expanding $R[k]$ by 2 in frequency. Expanding a sequence in frequency by 2 corresponds (by duality) to repeating the time sequence $r[n]$ twice, with an additional factor of 1/2. So over $0 \le n \le 1023$ we get two copies of $r[n]$, each scaled by 1/2:

$$y[n] = \begin{cases} \frac{1}{2}\Big(x[n] + x[n+512]\Big), & 0 \le n \le 511, \\ \frac{1}{2}\Big(x[n] + x[n-512]\Big), & 512 \le n \le 1023, \\ 0, & \text{otherwise.} \end{cases}$$

This matches choice A.

Problem 8.50

We have a 10–point sequence $x[n]$. Its z–transform is

$$X(z) = \sum_{n=0}^{9} x[n] z^{-n}.$$

The usual 10–point DFT samples this on the unit circle:

$$X[k] = X\left(e^{j2\pi k/10}\right).$$

We want to modify $x[n]$ so that the DFT of the new sequence $x_1[n]$ gives samples on the smaller circle of radius $\frac{1}{2}$ shown in Fig. P50–2. This means the samples should be

$$X_1[k] = X(z)\Big|_{z=\frac{1}{2}e^{j(2\pi k/10+\gamma/10)}}.$$

Using the form of $X(z)$,

$$X_1[k] = X\left(\tfrac{1}{2}e^{j(2\pi k/10+\gamma/10)}\right) = \sum_{n=0}^{9} x[n] \left(\tfrac{1}{2}e^{j(2\pi k/10+\gamma/10)}\right)^{-n}.$$

We want this to equal the 10–point DFT of a modified sequence $x_1[n]$:

$$X_1[k] = \sum_{n=0}^{9} x_1[n] W_{10}^{kn}, \qquad W_{10} = e^{-j2\pi/10}.$$

Matching terms, note that

$$W_{10}^{kn} = e^{-j2\pi kn/10}.$$

The expression we need to match is

$$\left(\tfrac{1}{2}e^{j(2\pi k/10+\gamma/10)}\right)^{-n} = \left(\tfrac{1}{2}\right)^{-n} e^{-j2\pi kn/10} e^{-j\gamma n/10}.$$

Thus

$$x_1[n]\, e^{-j2\pi kn/10} = x[n]\left(\tfrac{1}{2}\right)^{-n} e^{-j\gamma n/10} e^{-j2\pi kn/10}.$$

Comparing coefficients gives

$$x_1[n] = x[n]\left(\tfrac{1}{2}\right)^{-n} e^{-j\gamma n/10}. = x[n]\left(2\, e^{-j\gamma/10}\right)^{n}.$$

This sequence $x_1[n]$ produces the equally spaced samples of $X(z)$ on the radius–$\frac{1}{2}$ circle.

Problem 8.50

Given a 10–point sequence $x[n]$ whose z–transform is

$$X(z) = \sum_{n=0}^{9} x[n] z^{-n}.$$

The usual 10–point DFT samples this on the unit circle:

$$X[k] = X\left(e^{j2\pi k/10}\right).$$

We need to modify $x[n]$ so that the DFT of the new sequence $x_1[n]$ gives samples on the smaller circle of radius $\frac{1}{2}$ shown in Fig. P50–2. This means the samples should be

$$X_1[k] = X(z)\Big|_{z=\frac{1}{2}e^{j(2\pi k/10+\gamma/10)}}.$$

Using the form of $X(z)$,

$$X_1[k] = X\left(\tfrac{1}{2}e^{j(2\pi k/10+\gamma/10)}\right) = \sum_{n=0}^{9} x[n]\left(\tfrac{1}{2}e^{j(2\pi k/10+\gamma/10)}\right)^{-n}.$$

This should equal the 10–point DFT of a modified sequence $x_1[n]$:

$$X_1[k] = \sum_{n=0}^{9} x_1[n] W_{10}^{kn}, \qquad W_{10} = e^{-j2\pi/10}.$$

Matching terms, we see

$$W_{10}^{kn} = e^{-j2\pi kn/10}.$$

The expression we need to match is

$$\left(\tfrac{1}{2}e^{j(2\pi k/10+\gamma/10)}\right)^{-n} = \left(\tfrac{1}{2}\right)^{-n} e^{-j2\pi kn/10} e^{-j\gamma n/10}.$$

Thus

$$x_1[n]\, e^{-j2\pi kn/10} = x[n]\left(\tfrac{1}{2}\right)^{-n} e^{-j\gamma n/10} e^{-j2\pi kn/10}.$$

Comparing coefficients gives

$$x_1[n] = x[n]\left(\tfrac{1}{2}\right)^{-n} e^{-j\gamma n/10}. = x[n]\left(2\, e^{-j\gamma/10}\right)^{n}.$$

Exercise 5.1

We create a length–16 sinusoid

$$x[n] = \sin(\omega_0 n), \qquad \omega_0 = \frac{2\pi}{\sqrt{17}}.$$

We compute the 16–point DFT and call it $X[k]$. Then we zero pad $x[n]$ to lengths 32, 64, and 256 and compute the longer DFTs. The original 16 values of $X[k]$ appear exactly at corresponding frequency locations inside each longer DFT, and the remaining values in the longer transforms are interpolated values of the original spectrum.

```
Original 16 DFT samples:
  Columns 1 through 7

   0.4817 + 0.0000i   0.4939 + 0.1488i   0.5441 + 0.3652i   0.7404 + 0.9381i  -2.5203 - 7.2727i   0.0287 - 0.7333i   0.1631 - 0.3198i

  Columns 8 through 14

   0.2028 - 0.1344i   0.2128 + 0.0000i   0.2028 + 0.1344i   0.1631 + 0.3198i   0.0287 + 0.7333i  -2.5203 + 7.2727i   0.7404 - 0.9381i

  Columns 15 through 16

   0.5441 - 0.3652i   0.4939 - 0.1488i

Check that the same samples appear in X32, X64, X256 at expected indices:
  Columns 1 through 7

   0.4817 + 0.0000i   0.4939 + 0.1488i   0.5441 + 0.3652i   0.7404 + 0.9381i  -2.5203 - 7.2727i   0.0287 - 0.7333i   0.1631 - 0.3198i
   0.4817 + 0.0000i   0.4939 + 0.1488i   0.5441 + 0.3652i   0.7404 + 0.9381i  -2.5203 - 7.2727i   0.0287 - 0.7333i   0.1631 - 0.3198i
   0.4817 + 0.0000i   0.4939 + 0.1488i   0.5441 + 0.3652i   0.7404 + 0.9381i  -2.5203 - 7.2727i   0.0287 - 0.7333i   0.1631 - 0.3198i

  Columns 8 through 14

   0.2028 - 0.1344i   0.2128 + 0.0000i   0.2028 + 0.1344i   0.1631 + 0.3198i   0.0287 + 0.7333i  -2.5203 + 7.2727i   0.7404 - 0.9381i
   0.2028 - 0.1344i   0.2128 + 0.0000i   0.2028 + 0.1344i   0.1631 + 0.3198i   0.0287 + 0.7333i  -2.5203 + 7.2727i   0.7404 - 0.9381i
   0.2028 - 0.1344i   0.2128 + 0.0000i   0.2028 + 0.1344i   0.1631 + 0.3198i   0.0287 + 0.7333i  -2.5203 + 7.2727i   0.7404 - 0.9381i

  Columns 15 through 16

   0.5441 - 0.3652i   0.4939 - 0.1488i
   0.5441 - 0.3652i   0.4939 - 0.1488i
   0.5441 - 0.3652i   0.4939 - 0.1488i
```

```
N = 16;
w0 = 2*pi/sqrt(17);
n = 0:N-1;

x = sin(w0*n);       % length-16 signal
X16 = fft(x,16);     % 16-point DFT

% Zero-padded DFTs
X32  = fft(x,32);
X64  = fft(x,64);
X256 = fft(x,256);


disp('Original 16 DFT samples:')
disp(X16)

disp('Check that the same samples appear in X32, X64, X256 at expected indices:')
disp([X32(1:2:end); X64(1:4:end); X256(1:16:end)])
```

Exercise 5.2

This problem studies zero padding in the middle of a signal instead of at the end.

(a) We choose a real and even–symmetric test signal of odd length, for example $N = 21$. Because the signal is real and even, its DFT is real and even. This is verified by computing the DFT.

(b) We take the same signal and compute the 63–point DFT. Becaue the padding was done at the end, the symmetry is destroyed and the DFT is no longer purely real.

(c) We create a new zero–padded signal of length 63 using the middle–padding method:

1. Copy the first $\frac{1}{2}(N+1)$ samples of the original signal to the beginning. 2. Add $2N$ zeros. 3. Copy the last $\frac{1}{2}(N-1)$ samples to the end.

This preserves even symmetry of the original signal, and its 63–point DFT remains real and even. IN addition the interpolation property also holds.

```
Are DFT samples real for middle-padded case?
    1.7528

Compare original X[k] to downsampled longer DFT:
  Columns 1 through 7

  -0.0449 + 0.0000i -10.3818 - 1.5648i   2.0272 + 0.6253i  -0.0018 - 0.0009i  -0.0000 - 0.0000i   0.0001 + 0.0001i   0.0001 + 0.0001i
  -0.0449 + 0.0000i -10.3818 - 1.5648i   2.0272 + 0.6253i  -0.0018 - 0.0009i  -0.0000 - 0.0000i   0.0001 + 0.0001i   0.0001 + 0.0001i

  Columns 8 through 14

   0.0001 + 0.0001i   0.0000 + 0.0001i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0000i   0.0000 - 0.0000i   0.0000 - 0.0001i
   0.0001 + 0.0001i   0.0000 + 0.0001i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0000i   0.0000 - 0.0000i   0.0000 - 0.0001i

  Columns 15 through 21

   0.0001 - 0.0001i   0.0001 - 0.0001i   0.0001 - 0.0001i  -0.0000 + 0.0000i  -0.0018 + 0.0009i   2.0272 - 0.6253i -10.3818 + 1.5648i
   0.0001 - 0.0001i   0.0001 - 0.0001i   0.0001 - 0.0001i  -0.0000 + 0.0000i  -0.0018 + 0.0009i   2.0272 - 0.6253i -10.3818 + 1.5648i
```

```
N = 21;
m = -(N-1)/2 : (N-1)/2;
x = cos(0.3*m) + 0.2*cos(0.6*m);  % real and even
x = x(:).';                       % row vector

X21 = fft(x,21);

x_endpad = [x zeros(1,63-21)];
X63_end = fft(x_endpad,63);       % no longer purely real

%middle padding
L1 = (N+1)/2;
% take last (N-1)/2 samples
L2 = (N-1)/2;

x_mid = [ x(1:L1) zeros(1,2*N) x(end-L2+1:end) ];
X63_mid = fft(x_mid,63);          % symmetry preserved

% realness
disp('Are DFT samples real for middle-padded case?')
disp(max(abs(imag(X63_mid))))

% interp property
disp('Compare original X[k] to downsampled longer DFT:')
disp([X21; X63_mid(1:3:end)])
```