Gautham Anne
EE 359: Digital Signal Processing
11/02/2025

Homework ~~2~~ 5

PROBLEM 5.33

Given:

$$H(z) = \frac{(1-2z^{-1})(1-0.75z^{-1})}{z^{-1}(1-0.5z^{-1})} = \frac{(1-2z^{-1})(1-\frac{3}{4}z^{-1})}{z^{-1}(1-\frac{1}{2}z^{-1})}.$$

(a) Minimum-phase × all-pass. We want

$$H(z) = H_{\min 1}(z)\, H_{ap}(z)$$

with $H_{\min 1}(z)$ minimum phase (all zeros and poles inside $|z| < 1$) and $H_{ap}(z)$ unity-gain all-pass.
Step 1: put the inside-unit-circle factors into the minimum-phase part. Inside the unit circle we already have

$$\text{zero at } z = \tfrac{3}{4}, \qquad \text{pole at } z = \tfrac{1}{2}.$$

So a first try is

$$H^{(0)}_{\min 1}(z) = \frac{z-\frac{3}{4}}{z-\frac{1}{2}}.$$

The remaining factors (those outside $|z| = 1$ and at $z = 0$ or ∞) go to the all-pass part: zero at $z = 2$, pole at $z = 0$.
Step 2: make the second part actually all-pass. A 1-D all-pass has its zeros and poles in reciprocal pairs. We already have a zero at $z = 2$ (outside), so we must pair it with a pole at $z = \frac{1}{2}$ (inside). Likewise, we already have a pole at $z = 0$ (i.e. at ∞ in the $w = 1/z$ plane), so we must pair it with a zero at $z = \infty$ (i.e. a factor of z). The neat way to do this is to *move* the pole at $z = \frac{1}{2}$ from $H^{(0)}_{\min 1}(z)$ to the all-pass and then compensate in the minimum-phase part.

Concretely, take

$$H_{ap}(z) = \frac{z(z-2)}{z-\frac{1}{2}} = \frac{1-2z^{-1}}{(1-\frac{1}{2}z^{-1})\,z^{-1}}.$$

This has

- zeros at $z = 2$ and at ∞ (the factor z),
- poles at $z = \frac{1}{2}$ and at $z = 0$,

so the zeros/poles come in reciprocal pairs and $H_{ap}(e^{j\omega})$ has unit magnitude.

Then the minimum-phase part must be what is left:

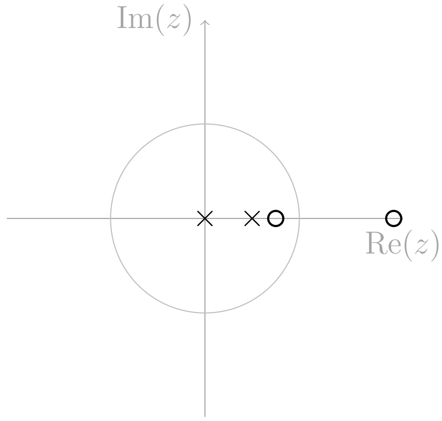
$$H_{\min 1}(z) = \frac{H(z)}{H_{ap}(z)} = \frac{(1-2z^{-1})(1-\frac{3}{4}z^{-1})}{z^{-1}(1-\frac{1}{2}z^{-1})} \cdot \frac{(1-\frac{1}{2}z^{-1})z^{-1}}{1-2z^{-1}} = 1 - \tfrac{3}{4}z^{-1}.$$

Equivalently,

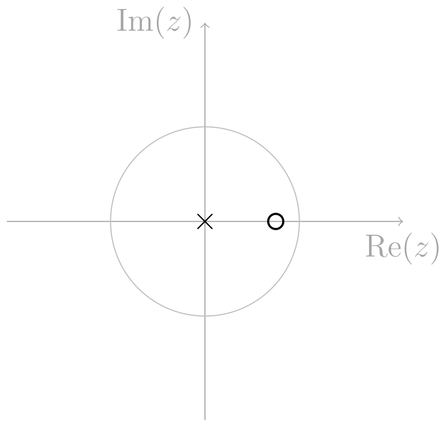
$$H_{\min 1}(z) = \frac{z-\frac{3}{4}}{z}.$$

This has a zero at $z = \frac{3}{4}$ and a pole at $z = 0$, both inside $|z| = 1$, so it is minimum phase.
Uniqueness. We never had a "free" choice: every time we introduced a pole/zero into the all-pass, the matching reciprocal factor was *forced*, and anything added to the all-pass would have to be cancelled in the minimum-phase part, but items outside the unit circle cannot be cancelled by a minimum-phase system. Hence, apart from an overall (nonzero) scale factor, the decomposition is unique.

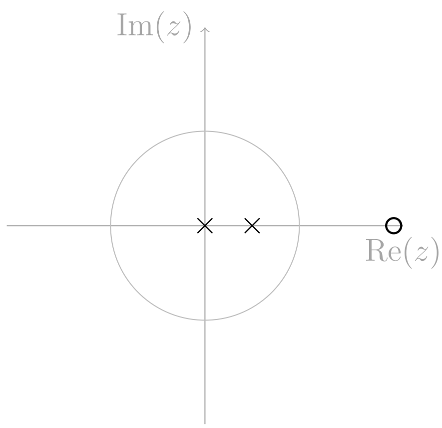
Pole–zero pictures. Original system:



Minimum-phase part $H_{\min 1}(z) = 1 - \frac{3}{4}z^{-1}$:



All-pass part $H_{ap}(z) = \dfrac{1 - 2z^{-1}}{(1 - \frac{1}{2}z^{-1})z^{-1}}$:



So the answer for part (a) is

$$\boxed{H_{\min 1}(z) = 1 - \tfrac{3}{4}z^{-1}, \qquad H_{ap}(z) = \frac{1 - 2z^{-1}}{(1 - \frac{1}{2}z^{-1})\, z^{-1}}}$$

and this decomposition is unique up to a constant gain.

(b) Minimum-phase × generalized linear-phase FIR. Now we want

$$H(z) = H_{\min 2}(z)\, H_{lp}(z)$$

where $H_{lp}(z)$ is a *linear-phase FIR*. That means $H_{lp}(z)$ must be FIR (so all its poles are at $z = 0$ or ∞) and the zeros must occur in reciprocal and conjugate-symmetric fashion to give linear phase.

Look again at the pole–zero pattern of $H(z)$:

- zero at $z = 2$ (outside),
- zero at $z = \frac{3}{4}$ (inside),
- pole at $z = \frac{1}{2}$ (inside),
- pole at $z = 0$.

A simple way to get a linear-phase FIR factor is to force the outside zero $z = 2$ to live in the FIR part together with its reciprocal $z = \frac{1}{2}$ so that the FIR part has symmetric zeros.

So take for the FIR part

$$H_{lp}(z) = z\left(1 - \tfrac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right) = z - 2.5 + z^{-1}.$$

This is a 3-tap, even-symmetric sequence, so it is linear phase. Its zeros are at $z = 2$ and $z = \frac{1}{2}$.

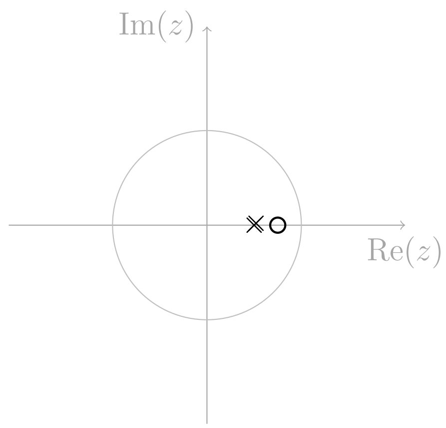
Then,

$$H_{\min 2}(z) = \frac{H(z)}{H_{lp}(z)} = \frac{(1 - 2z^{-1})(1 - \frac{3}{4}z^{-1})}{z^{-1}(1 - \frac{1}{2}z^{-1})} \cdot \frac{1}{z(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})} = \frac{1 - \frac{3}{4}z^{-1}}{(1 - \frac{1}{2}z^{-1})^2}.$$

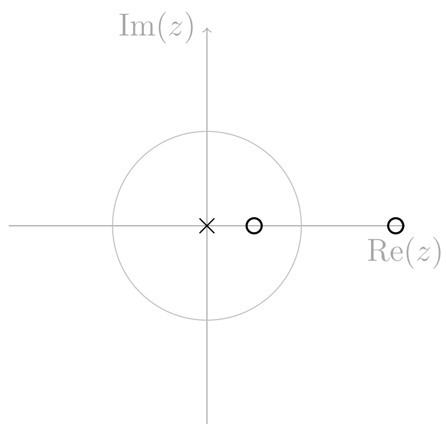
Equivalently,

$$H_{\min 2}(z) = \frac{z - \frac{3}{4}}{(z - \frac{1}{2})^2} = \frac{1 - \frac{3}{4}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)^2}.$$

This has a zero at $z = \frac{3}{4}$ and a (double) pole at $z = \frac{1}{2}$, all inside $|z| < 1$, so it is minimum phase.
Uniqueness. Exactly the same logic as in part (a): once FIR part be linear phase, any attempt to move a pole/zero between the two factors would either (i) break linear phase (because we'd have to reflect an inside-circle item to the outside) or (ii) break minimum phase (because we would need to keep an outside zero in the minimum-phase factor). Therefore the factorization is uniqe up to a scale.
Pole–zero pictures. Minimum-phase part $H_{\min 2}(z)$:



Linear-phase FIR part $H_{lp}(z) = z - 2.5 + z^{-1}$:



So the answer for part (b) is

$$\boxed{H_{\min 2}(z) = \frac{1-\frac{3}{4}z^{-1}}{(1-\frac{1}{2}z^{-1})^2}, \qquad H_{lp}(z) = z\Big(1-\tfrac{1}{2}z^{-1}\Big)\Big(1-2z^{-1}\Big) = z - 2.5 + z^{-1}}$$

and, again, the decomposition is unique up to a constant gain.

Problem 4.50

Determine whether system has a stable inverse system.

From the plotted $H(e^{j\omega})$ we see that the frequency response touches (and actually crosses) zero at some frequency ω_0 on the unit circle. That means the system has a zero on the unit circle, i.e.

$$H(e^{j\omega_0}) = 0 \quad \text{for some } \omega_0.$$

If we form the inverse system,

$$H_{\text{inv}}(e^{j\omega}) = \frac{1}{H(e^{j\omega})},$$

then at $\omega = \omega_0$ the inverse would require division by zero, which corresponds in the z-domain to the inverse system having a pole on the unit circle. A pole on the unit circle makes the inverse system unstable (its impulse response would not be absolutely summable). Therefore, the given system does not have a stable inverse system.

Problem 5.51

A causal LTI system has the system function

$$H(z) = \frac{(1-2z^{-1})(1+jz^{-1})(1+0.9z^{-1})}{(1-z^{-1})(1+0.7jz^{-1})(1-0.7jz^{-1})}.$$

(a) First write $H(z) = \dfrac{Y(z)}{X(z)}$ as a ratio of polynomials in z^{-1}.

$$\begin{aligned} H(z) &= \frac{(1-2z^{-1})(1+jz^{-1})(1+0.9z^{-1})}{(1-z^{-1})\Big(1+0.7jz^{-1}\Big)\Big(1-0.7jz^{-1}\Big)} \\ &= \frac{1-0.6z^{-1}-2.35z^{-2}-0.9z^{-3}}{1-z^{-1}+0.49z^{-2}-0.49z^{-3}} \\ &= \frac{Y(z)}{X(z)}. \end{aligned}$$

Hence

$$\Big(1-z^{-1}+0.49z^{-2}-0.49z^{-3}\Big)Y(z) = \Big(1-0.6z^{-1}-2.35z^{-2}-0.9z^{-3}\Big)X(z).$$

Taking the inverse z-transform gives the difference equation

$$\boxed{y[n]-y[n-1]+0.49\,y[n-2]-0.49\,y[n-3] = x[n]-0.6\,x[n-1]-2.35\,x[n-2]-0.9\,x[n-3].}$$

(b) Zeros:

$$z = 2, \quad z = -j, \quad z = -0.9.$$

Poles:

$$z = 1, \quad z = 0.7j, \quad z = -0.7j.$$

Since the system is *causal*, the ROC must be the outside of the outermost pole, so

$$\boxed{\text{ROC: } |z| > 1.}$$

(d)

(i) *"The system is stable."* False. For a causal LTI system to be stable, the ROC must include the unit circle. Here the ROC is $|z| > 1$, so $|z| = 1$ is not included. Hence the system is not BIBO–stable.

(ii) *"The impulse response approaches a constant for large n."* False. Because there is a pole on the unit circle at $z = 1$, the response does not settle to a finite constant in the BIBO–stable sense.
(iii) *"$|H(e^{j\omega})|$ has a peak at approximately $\omega = \pm\pi/4$."* False. From the form of the poles (at $\pm 0.7j$) the peaks occur nearer $\omega = \pm\pi/2$, not at $\pm\pi/4$.
(iv) *"The system has a stable and causal inverse."* False. The inverse would swap the poles and zeros. That would put a pole at $z = 2$ (outside) and at $z = -0.9$ (inside), so no single-sided ROC can include both and still contain the unit circle. Thus no inverse can be both causal and stable.

Problem 5.57

We have

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}, \qquad h[n] = h[M-n].$$

Split the sum into two equal halves (there is no middle sample now):

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=0}^{(M-1)/2} h[M-n]e^{-j\omega(M-n)} \\ &= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega M}e^{j\omega n} \\ &= e^{-j\omega M/2} \sum_{n=0}^{(M-1)/2} 2h[n]\cos\Big(\omega(M/2-n)\Big). \end{aligned}$$

Let

$$b[n] = 2h\Big(\frac{M+1}{2} - n\Big), \quad n = 1, \dots, \frac{M+1}{2},$$

then

$$H(e^{j\omega}) = e^{-j\omega(M/2)} \sum_{n=1}^{(M+1)/2} b[n]\cos\Big(\omega(n - \tfrac{1}{2})\Big).$$

Hence for Type II filters

$$A(e^{j\omega}) = \sum_{n=1}^{(M+1)/2} b[n]\cos\Big(\omega(n-\tfrac{1}{2})\Big), \qquad \alpha = \frac{M}{2}, \qquad \beta = 0.$$

Now $h[n] = -h[M-n]$ and $h[M/2] = 0$. Then

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\ &= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} + \sum_{n=0}^{M/2-1} h[M-n]e^{-j\omega(M-n)} \\ &= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} - \sum_{n=0}^{M/2-1} h[n]e^{-j\omega M}e^{j\omega n} \\ &= e^{-j\omega M/2} \sum_{n=0}^{M/2-1} 2j\, h[n]\sin\Big(\omega(M/2-n)\Big). \end{aligned}$$

Define

$$c[n] = 2h\Big(\frac{M}{2} - n\Big), \quad n = 1, \dots, \frac{M}{2},$$

so

$$H(e^{j\omega}) = e^{-j\omega(M/2)}e^{j\pi/2} \sum_{n=1}^{M/2} c[n]\sin(\omega n).$$

Thus for Type III filters

$$A(e^{j\omega}) = \sum_{n=1}^{M/2} c[n]\sin(\omega n), \qquad \alpha = \frac{M}{2}, \qquad \beta = \frac{\pi}{2}.$$

Problem 1.2: Impulse Response with `filter`

We are given the difference equation

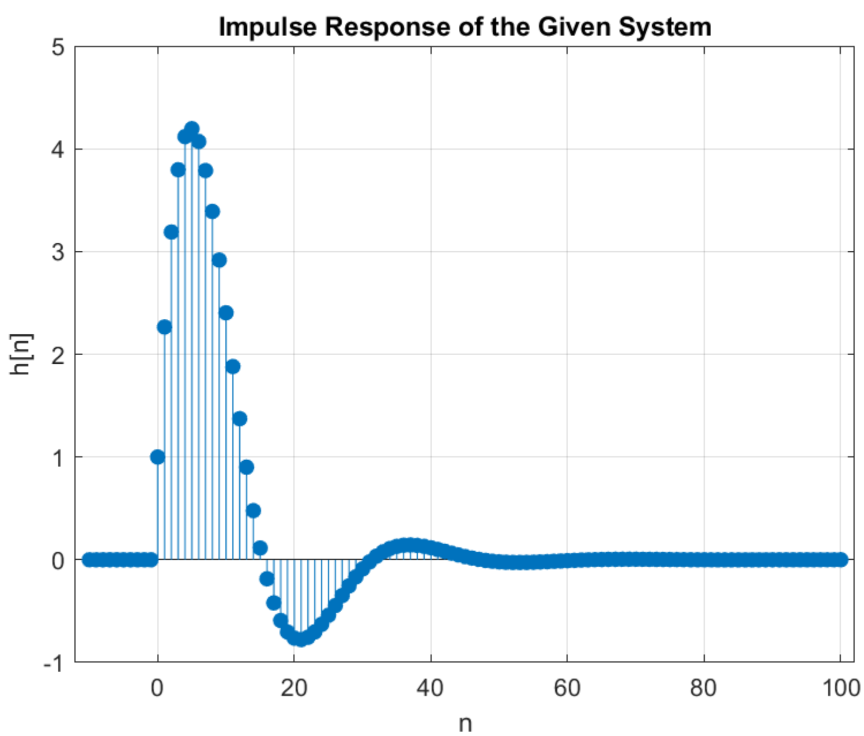
$$y[n] - 1.8\cos\left(\frac{\pi}{16}\right) y[n-1] + 0.81 y[n-2] = x[n] + \tfrac{1}{2}x[n-1].$$

The transfer function is

$$H(z) = \frac{1 + 0.5z^{-1}}{1 - 1.8\cos(\pi/16)z^{-1} + 0.81z^{-2}}.$$

The impulse response $h[n]$ is obtained using the `filter()` function in MATLAB.

```
clear; clc;
a = [1, -1.8*cos(pi/16), 0.81];
b = [1, 0.5];
n = -10:100;
x = (n==0);
h = filter(b, a, x);
stem(n, h, 'filled');
xlabel('n'); ylabel('h[n]');
title('Impulse Response');
grid on;
saveas(gcf, 'image.png');
```



Problem 3.1: Frequency Response with `freqz`

The given system has transfer function

$$H(z) = \frac{1 + 0.5z^{-1}}{1 - 1.8\cos\left(\frac{\pi}{16}\right) z^{-1} + 0.81z^{-2}}.$$

We can compute its frequency response using the `freqz` function in MATLAB, which evaluates $H(e^{j\omega})$ on the unit circle.

```
clear; clc;
a = [1, -1.8*cos(pi/16), 0.81];
b = [1, 0.5];

% Full frequency range (0 to 2 )
[H, w] = freqz(b, a, 512, 'whole');
figure;
subplot(2,1,1); plot(w, abs(H)); ylabel('|H(e^{j\omega})|');
subplot(2,1,2); plot(w, angle(H)); ylabel(' H(e^{j\omega})'); xlabel('  (rad/sample)');
sgtitle('Frequency Response (0 to 2 )');
saveas(gcf, 'image_full.png');

% Half range (0 to  )
[H_half, w_half] = freqz(b, a, 512);
figure;
subplot(2,1,1); plot(w_half, abs(H_half)); ylabel('|H(e^{j\omega})|');
subplot(2,1,2); plot(w_half, angle(H_half)); ylabel(' H(e^{j\omega})'); xlabel('  (rad/sample)'
sgtitle('Frequency Response (0 to  )');
saveas(gcf, 'image_half.png');
```

This script first computes the full $0 \leq \omega \leq 2\pi$ frequency response, then repeats for $0 \leq \omega \leq \pi$, which is sufficient for real-coefficient systems due to conjugate symmetry.

From the plots, the filter exhibits a strong low-frequency gain and attenuation at high frequencies, so it behaves a s a low pass filter.

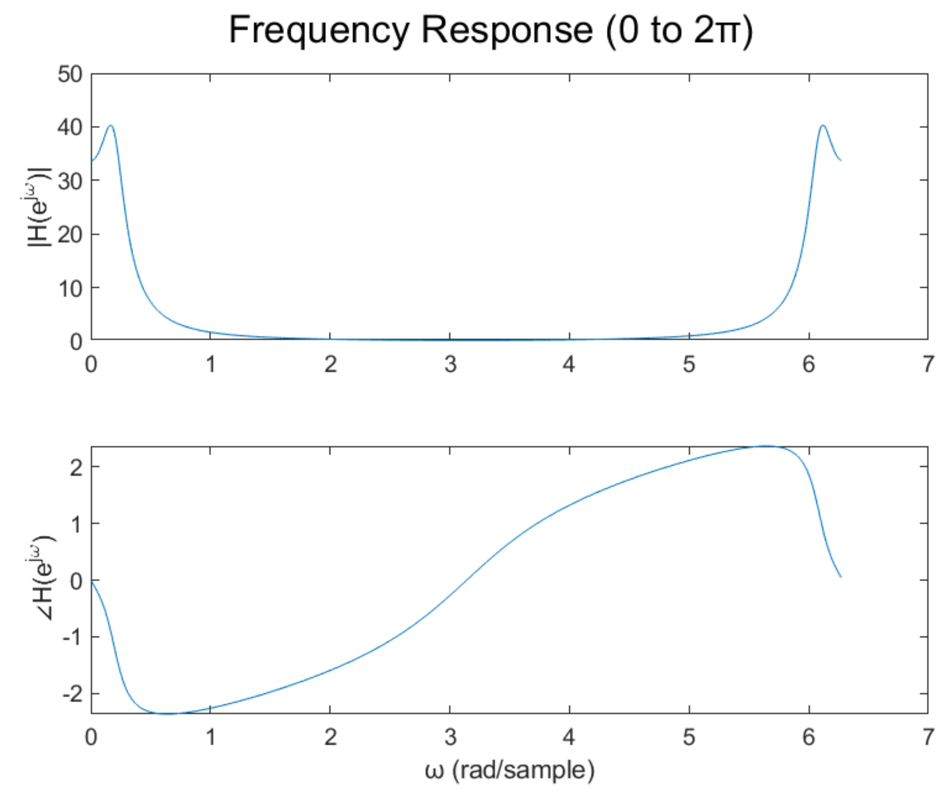


FIGURE 1. Magnitude and phase response of $H(e^{j\omega})$ for $0 \leq \omega \leq 2\pi$.

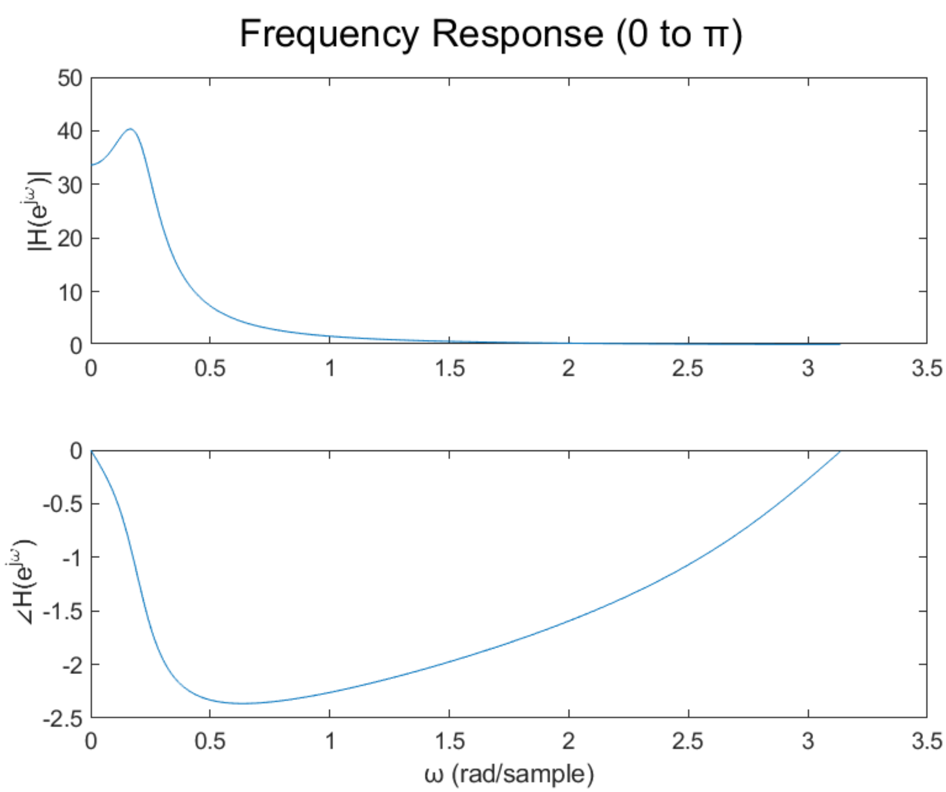


FIGURE 2. Magnitude and phase response of $H(e^{j\omega})$ for $0 \leq \omega \leq \pi$.