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EE 359: Digital Signal Processing
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Homework 3

Problem 3.3

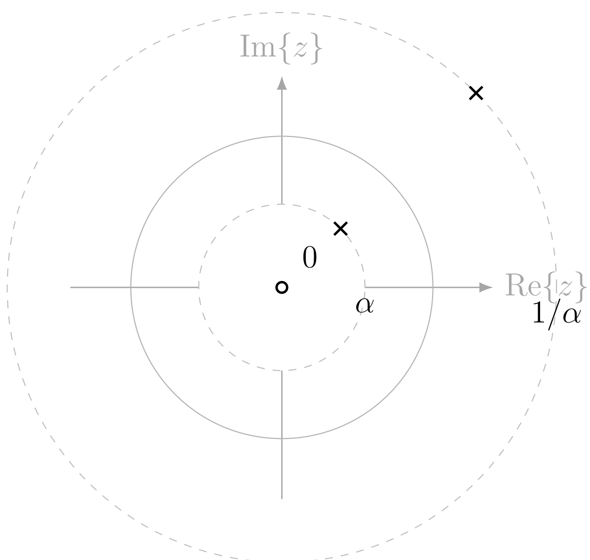
Question 1. (a) $x_a[n] = \alpha^{|n|},\ 0 < |\alpha| < 1.$

$$X_a(z) = \sum_{n=-\infty}^{\infty} \alpha^{|n|} z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n + \sum_{n=1}^{\infty} (\alpha z)^n = \frac{1}{1-\alpha z^{-1}} + \frac{\alpha z}{1-\alpha z}.$$

Combining terms,

$$\boxed{X_a(z) = \frac{z(1-\alpha^2)}{(z-\alpha)(1-\alpha z)}}, \qquad \boxed{\text{ROC: } |\alpha| < |z| < \frac{1}{|\alpha|}}.$$

Poles at $z = \alpha$ and $z = \alpha^{-1}$; a zero at $z = 0$.

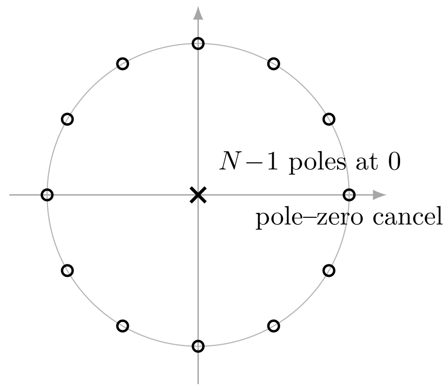


Question 2. (b) $x_b[n] = \begin{cases} 1, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$

$$X_b(z) = \sum_{n=0}^{N-1} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}} = \frac{z^N - 1}{z^{N-1}(z-1)} = \frac{1}{z^{N-1}} \sum_{k=0}^{N-1} z^k.$$

$$\boxed{X_b(z) = \frac{1-z^{-N}}{1-z^{-1}}}, \qquad \boxed{\text{ROC: } z \neq 0 \text{ (all } z \text{ except } 0)}.$$

Poles: a pole of order $N-1$ at $z = 0$. Zeros: the Nth roots of unity $\{e^{j2\pi m/N}\}_{m=0}^{N-1}$ with the zero at $z = 1$ canceled by the factor $(z-1)$ in the denominator, leaving $N-1$ zeros on the unit circle at $z = e^{j2\pi m/N},\ m = 1, \ldots, N-1$.



Question 3. (c) $x_c[n] = \begin{cases} n+1, & 0 \le n \le N-1, \\ 2N-1-n, & N \le n \le 2N-1, \\ 0, & \text{otherwise.} \end{cases}$ The triangular sequence is the convolution of two length-N rectangles:

$$x_c[n] = (x_b * x_b)[n].$$

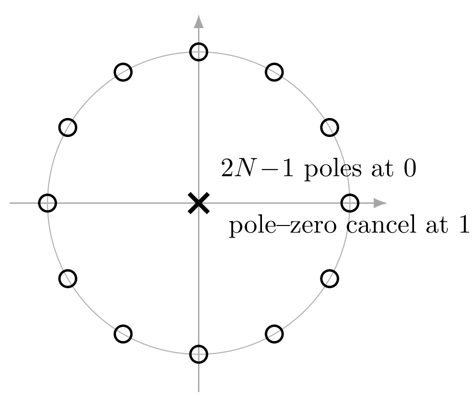
Therefore, in the z-domain,

$$X_c(z) = X_b(z)\,X_b(z) = \left(\frac{1-z^{-N}}{1-z^{-1}}\right)^2.$$

An equivalent form that makes pole order explicit is

$$\boxed{X_c(z) = z^{-1}\left(\frac{z^N-1}{z^{N-1}(z-1)}\right)^2 = \frac{(z^N-1)^2}{z^{2N-1}(z-1)^2}}, \quad \boxed{\text{ROC: } z \neq 0}.$$

Poles: a pole of order $2N-1$ at $z=0$. Zeros: double zeros at the Nth roots of unity; the double zero at $z=1$ cancels the factor $(z-1)^2$ in the denominator (no net feature at $z=1$).



Problem 3.30

System:

$$H(z) = \frac{1 - z^{-1}}{1 - 0.25\, z^{-2}} = \frac{1 - z^{-1}}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})}, \qquad \text{causal.}$$

Question 4. (a) Determine $y[n]$ for input $x[n] = u[n]$. With $X(z) = \mathcal{Z}\{u[n]\} = \frac{1}{1-z^{-1}}$ (ROC $|z| > 1$),

$$Y(z) = H(z)X(z) = \frac{1 - z^{-1}}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})} \cdot \frac{1}{1 - z^{-1}} = \frac{1}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})}.$$

Partial fractions:

$$\frac{1}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})} = \frac{\frac{1}{2}}{1 - 0.5z^{-1}} + \frac{\frac{1}{2}}{1 + 0.5z^{-1}}.$$

Inverse z-transform (causal sequences):

$$\boxed{y[n] = \tfrac{1}{2}(0.5)^n u[n] + \tfrac{1}{2}(-0.5)^n u[n]}.$$

Question 5. (b) Find the input $x[n]$ that yields $y[n] = \delta[n] - \delta[n-1]$. Here $Y(z) = 1 - z^{-1}$. Then

$$X(z) = \frac{Y(z)}{H(z)} = \frac{1 - z^{-1}}{\frac{1-z^{-1}}{1-0.25z^{-2}}} = 1 - 0.25\, z^{-2}.$$

Hence

$$\boxed{x[n] = \delta[n] - 0.25\, \delta[n-2]}.$$

Question 6. (c) Determine $y[n]$ for input $x[n] = \cos(0.5\pi n)$ for $-\infty < n < \infty$. For a real sinusoid $x[n] = \cos(\omega_0 n)$, the steady-state output is

$$y[n] = |H(e^{j\omega_0})| \cos\Big(\omega_0 n + \angle H(e^{j\omega_0})\Big).$$

Evaluate at $\omega_0 = \frac{\pi}{2}$:

$$H(e^{j\omega_0}) = \frac{1 - e^{-j\omega_0}}{1 - 0.25e^{-j2\omega_0}} = \frac{1 - (-j)}{1 - 0.25(-1)} = \frac{1 + j}{1.25}.$$

Thus

$$|H| = \frac{\sqrt{2}}{1.25} \approx 1.13, \qquad \angle H = \arg(1 + j) = \frac{\pi}{4}.$$

Therefore

$$\boxed{y[n] \approx 1.13 \cos\Big(0.5\pi n + \frac{\pi}{4}\Big).}$$

Problem 3.31(a)

We are given

$$X(z) = \frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}}.$$

Left-Sided Expansion. For a left-sided sequence, the ROC is inside the pole:

$$|z| < \tfrac{1}{3}.$$

Rewrite:

$$X(z) = \frac{z - \frac{1}{3}}{z + \frac{1}{3}} = 1 - \frac{\frac{2}{3}}{z + \frac{1}{3}}.$$

Expand about $z = 0$:

$$\frac{1}{z + \frac{1}{3}} = \frac{1}{\frac{1}{3}} \cdot \frac{1}{1 + 3z} = 3\sum_{k=0}^{\infty}(-3z)^k, \qquad |z| < \tfrac{1}{3}.$$

So

$$X(z) = 1 - 2\sum_{k=0}^{\infty}(-3z)^k.$$

Coefficient Identification. From the z-transform definition

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n},$$

we see that terms in positive powers of z correspond to negative-n coefficients.

Thus:

$$x[0] = -1, \qquad x[-k] = 2(-1)^{k-1}3^k, \quad k \geq 1, \qquad x[n] = 0,\ n > 0.$$

Thus, We get

$$\boxed{x[n] = -\delta[n] \; - \; 2\Big(-\tfrac{1}{3}\Big)^n \, u[-n-1],}$$

with ROC

$$\boxed{|z| < \tfrac{1}{3}.}$$

Problem 3.31(b,c)

Question 7. (b) **Partial fraction:** $X(z) = \dfrac{3}{z - \frac{1}{4} - \frac{1}{8}z^{-1}}$, with $x[n]$ stable.

Factor the denominator in z^{-1}:

$$X(z) = \frac{3z^{-1}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} = \frac{3z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{4}z^{-1})}.$$

Partial fractions:

$$\frac{3z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{4}z^{-1})} = \frac{4}{1 - \frac{1}{2}z^{-1}} - \frac{4}{1 + \frac{1}{4}z^{-1}}.$$

Poles at $z = \frac{1}{2}$ and $z = -\frac{1}{4}$. Stability $\Rightarrow$ ROC includes unit circle, so choose the cauesl ROC $|z| > \frac{1}{2}$. Hence

$$\boxed{x[n] = 4\left(\tfrac{1}{2}\right)^n u[n] - 4\left(-\tfrac{1}{4}\right)^n u[n], \qquad \text{ROC: } |z| > \tfrac{1}{2}.}$$

Question 8. (c) **Power series:** $X(z) = \ln(1 - 4z), \quad |z| < \frac{1}{4}$.

Use $\ln(1 - w) = -\sum_{i=1}^{\infty} \dfrac{w^i}{i}$ for $|w| < 1$ with $w = 4z$:

$$X(z) = -\sum_{i=1}^{\infty} \frac{(4z)^i}{i} = -\sum_{i=1}^{\infty} \frac{4^i}{i} z^i.$$

Since $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$, the term z^i corresponds to $n = -i$. Thus for $i \geq 1$,

$$x[-i] = -\frac{4^i}{i}, \qquad x[0] = 0, \qquad x[n] = 0 \ (n > 0).$$

Equivalently,

$$\boxed{x[n] = \frac{1}{n}\, 4^{-n}\, u[-n-1], \qquad \text{ROC: } |z| < \tfrac{1}{4} \ \text{(left-sided).}}$$

Problem 3.37

$$X(z) = \frac{z}{\left(z^2 - z + \frac{1}{2}\right)\left(z + \frac{1}{4}\right)}, \qquad \text{ROC: } |z| > \tfrac{3}{4}.$$

(The poles are at $z = \frac{1 \pm j}{2}$ and $z = -\frac{1}{4}$; there is a zero at $z = 0$.)

Using time-reversal and shift properties,

$$\mathcal{Z}\{x[-n]\} = X(z^{-1}), \qquad \mathcal{Z}\{x[-n+3]\} = z^{-3} X(z^{-1}).$$

Hence

$$\boxed{Y(z) = z^{-3}\, X\left(z^{-1}\right)} = z^{-3} \frac{z^{-1}}{\left(z^{-2} - z^{-1} + \frac{1}{2}\right)\left(z^{-1} + \frac{1}{4}\right)} = \frac{8/3}{z\,(2 - 2z + z^2)\,(\frac{3}{4} + z)}.$$

Poles, zeros, ROC. From the last form,

$$\text{Poles at } \boxed{z = 0,\ z = -\tfrac{3}{4},\ z = 1 \pm j}; \qquad \text{zeros at infinity (none finite).}$$

Because $x[n]$ is causal, $x[-n+3]$ is left-sided $\Rightarrow$ the ROC lies inside the smallest circle that excludes the outermost pole but outside the origin:

$$\boxed{\text{ROC for } Y(z): \quad 0 < |z| < \tfrac{4}{3}.}$$

Problem 3.45

Given

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + 2^n u[-n-1], \qquad y[n] = 6\left(\tfrac{1}{2}\right)^n u[n] - 6\left(\tfrac{3}{4}\right)^n u[n].$$

Question 9. (a) Find $H(z)$, its poles/zeros, and the ROC. Use $\mathcal{Z}\{a^n u[n]\} = \dfrac{1}{1-az^{-1}}$ (ROC $|z| > |a|$) and $\mathcal{Z}\{a^n u[-n-1]\} = -\dfrac{1}{1-az^{-1}}$ (ROC $|z| < |a|$). Hence

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}}, \qquad \text{ROC: } \tfrac{1}{2} < |z| < 2,$$

$$Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}}, \qquad \text{ROC: } |z| > \tfrac{3}{4}.$$

Therefore

$$H(z) = \frac{Y(z)}{X(z)} = \boxed{\frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}}$$

with ROC inherited from $Y(z)$ (and containing the unit circle for stability):

Poles/zeros: pole at $z = \frac{3}{4}$, zero at $z = 2$.

Question 10. (b) Find the impulse response $h[n]$. From $H(z) = \dfrac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}} = \dfrac{1}{1-\frac{3}{4}z^{-1}} - \dfrac{2z^{-1}}{1-\frac{3}{4}z^{-1}}$,

$$\boxed{h[n] = \left(\tfrac{3}{4}\right)^n u[n] - 2\left(\tfrac{3}{4}\right)^{n-1} u[n-1].}$$

Question 11. (c) Write the difference equation. Multiply both sides by $1 - \frac{3}{4}z^{-1}$ and invert:

$$(1 - \tfrac{3}{4}z^{-1})Y(z) = (1-2z^{-1})X(z) \iff \boxed{y[n] - \tfrac{3}{4}y[n-1] = x[n] - 2x[n-1].}$$

Question 12. (d) Is the system stable? Is it causal? Yes. The ROC $|z| > \frac{3}{4}$ includes the unit circle $\Rightarrow$ BIBO stability. Also $h[n] = 0$ for $n < 0 \Rightarrow$ causal.