Gautham Anne
EE 359: Electronic Properties of Materials
October 7, 2025

Homework 2

Problem 1

Which of the following discrete-time signals are eigenfunctions of stable, linear time-invariant (LTI) systems? Explain why.

Question 1.

1) $x_1[n] = e^{j\pi n/3} + e^{j\pi n/4}$

   Each term $e^{j\pi n/3}$ and $e^{j\pi n/4}$ is individually an eigenfunction of an LTI system. However, their sum is not a single eigenfunction, because the output would be

$$y[n] = H\left(e^{j\pi/3}\right) e^{j\pi n/3} + H\left(e^{j\pi/4}\right) e^{j\pi n/4},$$

   which is a linear combination of two exponentials, not a scaled version of $x_1[n]$. Thus, $x_1[n]$ is **not an eigenfunction**.

2) $x_2[n] = \sin\left(\frac{\pi}{3}n\right)$

   Using Euler's identity:

$$\sin\left(\frac{\pi}{3}n\right) = \frac{1}{2j}\left(e^{j\pi n/3} - e^{-j\pi n/3}\right).$$

   Each exponential is an eigenfunction, but their linear combination is not a single exponential. Thus $x_2[n]$ is **not an eigenfunction**.

3) $x_3[n] = 3^n u[n]$

   This can be written as

$$x_3[n] = \left(e^{\ln(3)}\right)^n u[n].$$

   The exponential part $e^{(\ln 3)n}$ looks like an eigenfunction, but the presence of $u[n]$ truncates it to $n \geq 0$. Because of the step function, the output of an LTI system will not remain a scaled version of $x_3[n]$. Hence $x_3[n]$ is **not an eigenfunction**.

4) $x_4[n] = \left(\frac{1}{2}\right)^n e^{j\pi n/5}$

   We can rewrite:

$$x_4[n] = \left(\frac{1}{2}e^{j\pi/5}\right)^n.$$

   This is a complex exponential with base $re^{j\omega}$ where $r = \frac{1}{2}$. General LTI systems have eigenfunctions of the form λ^n with complex λ. Therefore, $x_4[n]$ **is an eigenfunction**.

Only (d) is an eigenfunction of an LTI system.

Problem: O/S 2.26

For each system, pick the strongest valid conclusion from: (i) must be LTI and uniquely specified; (ii) must be LTI but not uniquely specified; (iii) could be LTI and, if so, uniquely specified; (iv) could be LTI but not uniquely specified; (v) could not possibly be LTI. For any case where you choose (i) or (iii), give $h[n]$.

System A. Input: $x[n] = (1/2)^n \rightarrow$ Output: $(1/4)^n$.

For an LTI system, λ^n is an eigenfunction: $T\{\lambda^n\} = H(\lambda)\lambda^n$. Since $(1/2)^n$ would have to map to a *scaled* $(1/2)^n$ (not $(1/4)^n$), the mapping violates the eigenfunction property.

Conclusion: (v) The system could not possibly be LTI.

System B. Input: $x[n] = \cos(\pi n/3) \;\rightarrow\;$ Output: $3j\,\sin(\pi n/3)$.

An LTI system excited by a sinusoid at frequency ω_0 produces a sinusoid at the *same* ω_0 with amplitude/phase change. The given input-output pair is consistent with some complex frequency response at $\omega_0 = \pi/3$, but it does not determine $H(e^{j\omega})$ for other ω.

$$\boxed{\text{Conclusion: (iv) The system could be LTI, but is not uniquely determined.}}$$

System C. Input: $x[n] = \delta[n+1] + \frac{1}{2}\,\delta[n]$, Output: $y[n] = \left(\frac{1}{3}\right)^n u[n]$.

If the system is LTI, then with $x[n] = \delta[n+1] + \frac{1}{2}\delta[n]$ we have

$$y[n] = h[n+1] + \tfrac{1}{2}\,h[n].$$

Equivalently in the z-domain:

$$X(z) = z^{-1} + \tfrac{1}{2}, \qquad Y(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}, \qquad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{\left(1 - \frac{1}{3}z^{-1}\right)\left(z^{-1} + \frac{1}{2}\right)}.$$

Partial fractions in $r = z^{-1}$:

$$\frac{1}{(1 - \frac{1}{3}r)(r + \frac{1}{2})} = \frac{\frac{2}{7}}{1 - \frac{1}{3}r} + \frac{\frac{6}{7}}{r + \frac{1}{2}}.$$

Taking inverse z-transforms (with the ROC matching $Y(z)$, i.e., right-sided for the first term and left-sided for the second):

$$h[n] = \frac{2}{7}\left(\tfrac{1}{3}\right)^n u[n] \;-\; \frac{6}{7}\,(-\tfrac{1}{2})^{n-1} u[-n-1].$$

Thus the information uniquely specifies an LTI system (its $H(z)$ is fixed by X and Y), but $h[n]$ is two-sided (noncausal) and not absolutely summable (unstable).

$$\boxed{\text{Conclusion: (iii) The system could be LTI and, if so, is uniquely specified, with}} \boxed{h[n] = \frac{2}{7}\left(\frac{1}{3}\right)^n u[n] - \frac{6}{7}\,(-\tfrac{1}{2})^{n-}}$$

Problem 54

Given $h_1[n] = \beta\,\delta[n-1]$ and $h_2[n] = \alpha^n u[n]$ with the adder before h_2.

(a) Impulse response $h[n]$ of the overall system. From the diagram, the effective input to h_2 is $x[n] + (x * h_1)[n] = x[n] + \beta x[n-1]$, so the overall impulse response is

$$h[n] = (\delta[n] + h_1[n]) * h_2[n] = h_2[n] + h_1[n] * h_2[n] = \alpha^n u[n] + \beta\,\alpha^{n-1} u[n-1].$$

(b) Frequency response $H(e^{j\omega})$. Using $\sum_{n=0}^{\infty}(\alpha e^{-j\omega})^n = \frac{1}{1-\alpha e^{-j\omega}}$ (for $|\alpha| < 1$) and the time shift:

$$\mathcal{F}\{\alpha^n u[n]\} = \frac{1}{1 - \alpha e^{-j\omega}}, \qquad \mathcal{F}\{\alpha^{n-1} u[n-1]\} = \frac{e^{-j\omega}}{1 - \alpha e^{-j\omega}}.$$

Hence

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} + \beta\,\frac{e^{-j\omega}}{1 - \alpha e^{-j\omega}} = \boxed{\frac{1 + \beta e^{-j\omega}}{1 - \alpha e^{-j\omega}}}.$$

(c) Difference equation relating $y[n]$ and $x[n]$. Since $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$,

$$(1 - \alpha e^{-j\omega})Y(e^{j\omega}) = (1 + \beta e^{-j\omega})X(e^{j\omega}).$$

Taking inverse Fourier transforms gives

$$\boxed{y[n] - \alpha\,y[n-1] = x[n] + \beta\,x[n-1].}$$

(d) Causality and stability. From (a), $h[n] = 0$ for $n < 0$; therefore the system is **causal**. BIBO stability requires $h[n] \in \ell^1$, which holds iff the geometric terms decay:

$$\sum_{n=0}^{\infty} \left|\alpha^n\right| < \infty \quad \Longleftrightarrow \quad \boxed{|\alpha| < 1}.$$

(For any finite β, the second term also decays under the same condition.)

Problem 55: Properties from $x[n]$ in Fig. P55

Let $X(e^{j\omega}) = \sum_n x[n]e^{-j\omega n}$.

(a) $X(e^{j\omega})\Big|_{\omega=0}$**.**

$$X(e^{j\cdot 0}) = \sum_n x[n] = \boxed{6}.$$

(b) $X(e^{j\omega})\Big|_{\omega=\pi}$**.**

$$X(e^{j\pi}) = \sum_n x[n]e^{-j\pi n} = \sum_n x[n](-1)^n = \boxed{2}.$$

(d) $\int_{-\pi}^{\pi} X(e^{j\omega})\,d\omega$**.** Using the DTFT identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega = x[n],$$

set $n = 0$:

$$\int_{-\pi}^{\pi} X(e^{j\omega})\,d\omega = 2\pi\,x[0] = \boxed{4\pi}.$$

(e) Signal with Fourier transform $X(e^{-j\omega})$. Let $Y(e^{j\omega}) = X(e^{-j\omega})$. Then

$$Y(e^{j\omega}) = \sum_n x[n]e^{+j\omega n} = \sum_n x[-n]e^{-j\omega n}$$

so the corresponding time signal is

$$\boxed{y[n] = x[-n]}.$$

That is, the sketch is the *time-reversed* version of $x[n]$ (mirror the stems of Fig. P55 about $n = 0$).

Problem: O/S 2.85

Let $X(e^{j\omega})$ be the 2π-periodic triangular spectrum

$$X(e^{j\omega}) = \begin{cases} 1 - \dfrac{|\omega|}{3\pi/4}, & |\omega| \le 3\pi/4, \\ 0, & 3\pi/4 < |\omega| \le \pi, \end{cases} \qquad \text{and periodic with } 2\pi.$$

We form, as in O/S 2.85:

$$y_s[n] = \begin{cases} x[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}, \qquad y_d[n] = x[2n], \qquad y_e[n] = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}.$$

(a) Sampler $y_s[n]$. Using $y_s[n] = \frac{1}{2}\left(1 + e^{j\pi n}\right)x[n]$,

$$\boxed{Y_s(e^{j\omega}) = \tfrac{1}{2}\Big[X(e^{j\omega}) + X\Big(e^{j(\omega+\pi)}\Big)\Big].}$$

(b) Compressor (downsample by 2) $y_d[n] = x[2n]$. Standard aliasing formula:

$$\boxed{Y_d(e^{j\omega}) = \tfrac{1}{2}\Big[X\Big(e^{j\omega/2}\Big) + X\Big(e^{j(\omega/2+\pi)}\Big)\Big]} \quad \text{(equivalently } Y_d(e^{j\omega}) = Y_s\Big(e^{j\omega/2}\Big)\text{)}.$$

(c) Expander (upsample by 2) $y_e[n]$. Upsampling property:

$$\boxed{Y_e(e^{j\omega}) = X(e^{j2\omega}).}$$