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EE 359: Digital Signal Processing
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Homework 1b

Exercise 2.48

Question 1. (a) Determine whether the system T could be linear. From the figure, observe that

$$x_1[n] = x_2[n] + x_3[n+4].$$

If $T\{\cdot\}$ were linear, then

$$T\{x_1[n]\} = T\{x_2[n]\} + T\{x_3[n+4]\} = y_2[n] + y_3[n+4].$$

But $y_1[n] \neq y_2[n] + y_3[n+4]$. Thus the system violates superposition and is

$$\boxed{\text{not linear.}}$$

Question 2. (b) If the input $x[n]$ to the system is $\delta[n]$, what is the system response $y[n]$? Notice that $x_3[n] = \delta[n-4]$, i.e.

$$\delta[n] = x_3[n+4].$$

By time invariance,

$$T\{\delta[n]\} = T\{x_3[n+4]\} = y_3[n+4].$$

Since $y_3[n] = 3\delta[n+2] + 2\delta[n+1]$, shifting by $+4$ gives

$$\boxed{y[n] = 3\delta[n+6] + 2\delta[n+5].}$$

Question 3. (c) What are all possible inputs $x[n]$ for which the response of the system T can be determined from the given information alone? The system is time-invariant but not linear. Therefore, we cannot form $\delta[n]$ as linear combinations of the given inputs. The only responses we can determine are for inputs that are time-shifted versions of the given $x_i[n]$. Thus,

$$\boxed{\text{Only shifted versions of the given inputs } x_1[n], x_2[n], x_3[n].}$$

Exercise 2.49

Question 4. (a) Determine whether the system L could be time invariant. We attempt to form an impulse as a linear combination of the given inputs:

$$\delta[n] = \tfrac{1}{2}x_1[n] - \tfrac{1}{2}x_2[n] + x_3[n].$$

Since L is linear,

$$L\{\delta[n]\} = \tfrac{1}{2}y_1[n] - \tfrac{1}{2}y_2[n] + y_3[n].$$

Now, consider a shifted impulse:

$$\delta[n-1] = -\tfrac{1}{2}x_1[n] + \tfrac{1}{2}x_2[n],$$

so

$$L\{\delta[n-1]\} = -\tfrac{1}{2}y_1[n] + \tfrac{1}{2}y_2[n].$$

Comparing $L\{\delta[n-1]\}$ with the shifted version of $L\{\delta[n]\}$, we find they are not equal. Hence

$$\boxed{\text{The system is not time-invariant.}}$$

Question 5. (b) If the input $x[n]$ to the system L is $\delta[n]$, what is the system response $y[n]$? From above,

$$\delta[n] = \tfrac{1}{2}x_1[n] - \tfrac{1}{2}x_2[n] + x_3[n].$$

Therefore,

$$h[n] = L\{\delta[n]\} = \tfrac{1}{2}y_1[n] - \tfrac{1}{2}y_2[n] + y_3[n].$$

Expanding each:

$$\begin{aligned} y_1[n] &= -\delta[n+1] + 3\delta[n] + 3\delta[n-1] + \delta[n-3], \\ y_2[n] &= -\delta[n+1] + \delta[n] - 3\delta[n-1] - \delta[n-3], \\ y_3[n] &= 2\delta[n+2] + \delta[n+1] - 3\delta[n] + 2\delta[n-2]. \end{aligned}$$

Combine:

$$h[n] = 2\delta[n+2] + \delta[n+1] - 2\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3].$$

Thus the impulse response is

$$\boxed{h[n] = 2\delta[n+2] + \delta[n+1] - 2\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3].}$$

Exercise 2.51

Question 6. (a) Determine whether the system is time invariant. Test with a shift. Let $x_1[n] = \delta[n]$. Using the rule $y[n] - a\,y[n-1] = x[n]$ and $y[0] = 1$:

$$y_1[0] = 1, \qquad y_1[1] - a\,y_1[0] = x_1[1] = 0 \;\Rightarrow\; y_1[1] = a.$$

Now shift the input: $x_2[n] = \delta[n-1]$. Then

$$y_2[0] = 1, \qquad y_2[1] - a\,y_2[0] = x_2[1] = 1 \;\Rightarrow\; y_2[1] = a + 1.$$

If the system were time invariant, we would have $y_2[n] = y_1[n-1]$, so in particular $y_2[1]$ would equal $y_1[0] = 1$, but $y_2[1] = a + 1 \neq 1$ for general a. Hence

$$\boxed{\text{the system is not time invariant.}}$$

Question 7. (b) Determine whether the system is linear. Linearity requires homogeneity and additivity. A single counterexample suffices. Take the *zero input* $x[n] \equiv 0$. From the given rules,

$$y[0] = 1, \qquad y[n] - a\,y[n-1] = 0 \;\Rightarrow\; y[n] = a^n \quad (\text{for } n \geq 0).$$

Now scale the input by 2: $\tilde{x}[n] = 2x[n] \equiv 0$. The output is unchanged: $\tilde{y}[n] = a^n \neq 2\,a^n = 2y[n]$ (already violated at $n = 0$ where $\tilde{y}[0] = 1 \neq 2$). Thus homogeneity fails, so

$$\boxed{\text{the system is not linear.}}$$

Question 8. (c) Assume the difference equation (property 1) is the same, but $y[0] = 0$ is specified. Does this change your answer to either part (a) or part (b)? With $y[0] = 0$ fixed and $y[n] - a\,y[n-1] = x[n]$, the mapping becomes linear. For any inputs x_1, x_2 and scalars α, β, let y_1, y_2 be the corresponding outputs. Define $x_3 = \alpha x_1 + \beta x_2$ and y_3 its output. Then for $n \geq 0$,

$$\begin{aligned} y_3[n] - a\,y_3[n-1] &= x_3[n] = \alpha x_1[n] + \beta x_2[n], \\ (\alpha y_1[n] + \beta y_2[n]) - a(\alpha y_1[n-1] + \beta y_2[n-1]) &= \alpha x_1[n] + \beta x_2[n], \end{aligned}$$

and with the *zero initial condition* $y_3[0] = \alpha y_1[0] + \beta y_2[0] = 0$, the solution is unique, hence $y_3[n] = \alpha y_1[n] + \beta y_2[n]$ for all n. Therefore the system is

$$\boxed{\text{linear when } y[0] = 0.}$$

Time invariance, however, still fails. Repeating the shift test with $y[0] = 0$: for $x_1[n] = \delta[n]$, $y_1[0] = 0$, $y_1[1] = a + 1$; for $x_2[n] = \delta[n-1]$, $y_2[0] = 0$, $y_2[1] = a$. Since $y_2[1] \neq y_1[0]$, we again violate $y_2[n] = y_1[n-1]$. Hence

$$\boxed{\text{the system remains not time invariant.}}$$